EXISTENCE OF SOLUTIONS OF SINGULAR NONLINEAR SECOND-ORDER BOUNDARY VALUE PROBLEMS

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Abstract.

This study investigates the existence of solutions of singular nonlinear second-order boundary value problems. The existence of solutions is further developed based on the notion given by O’Regan (World Scientific Press, Singapore, 1994). It is proved that there is a solution for the singular non-linear second-order boundary value problems.

Keywords: Existence of solution, Singular nonlinear boundary value problems

1 Introduction

This paper deals with the existence of solutions of boundary value problems of the following types:

\[
\frac{1}{p(t)} (p(t)y''(t))' + f(t, y(t), p(t)y'(t)) = 0 \quad 0 < t < 1
\]

\[
y(1) = \lim_{t \to 0^+} p(t)y'(t) = 0
\]

where \( \lim_{y \to 0^+} f(t, y(t), p(t)y'(t)) = \infty \), \( p \in C[0,1] \cap C^1(0,1) \) with \( p > 0 \) on \( (0,1) \),

\( p(0) = 0 \) and \( \int_0^1 \frac{dt}{p(t)} = \infty \).
Problems of the form (*) occur during the search of positive radially symmetric solutions to certain nonlinear partial differential equations; for a more detailed discussion see [2,3,5]. The problems of the form (*) for $p \equiv 1$ have been considered by many authors with the most advanced results [5,8,9,10]. When $p \equiv t^{n-1}$ with $n \geq 2$ problems of the form (*) have been discussed in [4,5]. More existence principles of the general problem (*) have been given in [7]. In this paper, we will show an alternative existence theorem as an extension of the idea given in [10].

Remark The results of this paper also hold if
\[ \int_0^1 \frac{dt}{p(t)} < \infty \]

2 Existence Principles

We consider the “mixed” two-point boundary value problem

\[
\begin{align*}
\frac{1}{p}(py')' + f(t,y,py') &= 0 \quad 0 < t < 1 \\
\lim_{t \to 0^+} p(t)y'(t) &= 0, \quad y(1) = d
\end{align*}
\]

(2.1)

By a solution, we mean a function $y \in C[0,1] \cap C^1(0,1) \cap C^2(0,1)$ with $py' \in C[0,1]$ which satisfies the differential equation on $(0,1)$ and stated boundary condition in (2.1).

Associated with (2.1), we have a family of problems

\[
\begin{align*}
\frac{1}{p}(py')' + \lambda f(t,y,py') &= 0 \quad 0 < t < 1 \\
\lim_{t \to 0^+} p(t)y'(t) &= 0, \quad y(1) = d
\end{align*}
\]

(2.2)

where $0 < \lambda < 1$.

Theorem 2.1 (Nonlinear Alternative)

Let $C$ be a convex subset of a normed linear space $E$, $U$ be an open subset of $C$ and $\overline{U}$ and $\partial U$ be the closure of $U$ in $C$ and the boundary of $U$ in $C$. Let $F: \overline{U} \to C$ be a compact map with $p^* \in U$. Then either

i) $F$ has a fixed point in $\overline{U}$; or

ii) there is a point $x \in \partial U$ and $\lambda \in (0,1)$ such that $x = \lambda Fx + (1 - \lambda)p^*$ [7].
Theorem 2.2

\[ f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ is continuous} \quad (2.3) \]

\[ \begin{cases}
  p \in \mathbb{C}[0,1] \cap \mathbb{C}^1(0,1) \text{ together with } p > 0 \text{ on } (0,1) \\
  p(0) = 0 \text{ and } \int_0^1 \frac{dt}{p(t)} = \infty 
\end{cases} \quad (2.4) \]

\[ \int_0^1 p(t)dt < \infty, \quad \int_0^1 \frac{1}{p(t)} \int_0^s p(s)dsdt < \infty \quad (2.5) \]

In addition suppose there is a constant \( M \), independent of \( \lambda \), with

\[ \text{Max} \left\{ \sup_{[0,1]} |y(t)|, \sup_{[0,1]} |p(t)y'(t)| \right\} = \|y\| \leq M \]

Then (2.1) has at least one solution \( y \) in \( C[0,1] \cap C^1(0,1] \cap C^2(0,1) \) with \( py' \in C[0,1] \) [8,10]

Corollary 2.3

Suppose the conditions of Theorem 2.2 are satisfied. Assume \( p(0) \neq 0 \) or if \( p(0) = 0 \) then \( \lim_{t \to 0^+} (p(t)/p'(t)) \) exists. Then (2.1) has at least one solution in \( C^1[0,1] \cap C^2(0,1) \). If in addition \( p(0) \neq 0 \) or if \( p(0) = 0 \) and \( \lim_{t \to 0^+} (p(t)/p'(t)) = 0 \) then \( y'(0) = 0 \) for any solution \( y \in C^1[0,1] \cap C^2(0,1) \) to (2.1) [7].

Many mathematical models give rise to singular boundary value problems of the form

\[ \begin{cases}
  \frac{1}{p} (py')' + \Phi(t)y' = 0 & 0 < t < 1 \\
  \lim_{t \to 0^+} p(t)y'(t) = 0, \quad y(1) = 0
\end{cases} \]

with \( \Phi \in C[0,1], \; \Phi \geq 0 \) on \([0,1], \; p \in C[0,1] \cap C^1(0,1] \) with \( p > 0 \) on \([0,1]\) and \( \gamma > 0 \).

This study presents existence results for the general second order boundary value problems

\[ \begin{cases}
  \frac{1}{p} (py')' + f(t, y, py') = 0 & 0 < t < 1 \\
  \lim_{t \to 0^+} p(t)y'(t) = 0, \quad y(1) = 0 \\
  \lim_{y \to 0^+} f(t, y, py') = \infty
\end{cases} \quad (2.6) \]

with \( f : [0,1] \times (0,\infty) \times (-\infty,\infty) \to (0,\infty) \) continuous and \( p \in C[0,1] \cap C^1(0,1] \) with \( p > 0 \) on \((0,1)\).
Theorem 2.4

Suppose (2.4) is satisfied and in addition assume

\[ \int_0^1 p(t) \int_{p(s)}^t ds \, dt < \infty \]

\[ \|y\|^2 \leq \|y\|^2 \int_0^1 p(t) \int_{p(s)}^t ds \, dt \]

\[ \text{(2.7)} \]

for all function \( y \in \mathbb{D} = \{ w \in C[0,1] : w \text{ and } pw' \text{ are absolutely continuous on } [0,1] \}

and \( \frac{1}{p} (pw')' \in L_p^2 [0,1] \) with \( \lim_{t \to 0^+} p(t)w'(t) = 0 \), \( w(1) = 0 \}

where \( \|y\|^2 = \int_0^1 py^2 \, dt \) and \( \|y\|^2 = \int_0^1 p(y)^2 \, dt \) [8].

Theorem 2.5

Let \( n \in \mathbb{N}^+ = \{ 1, 2, \ldots \} \) and suppose \( \lim_{t \to 0^+} p(t)w'(t) = 0 \), \( w(1) = 1/n \) and (2.4) and (2.5) are satisfied

i) Let \( y(t) > 1/n \) for \( t \in (0,1) \) and \( 0 < \gamma < 1 \) then

\[ \|y\|^2 \leq \|y\|^2 \int_0^1 p(t) \int_{p(s)}^t ds \, dt + 2 \int_0^1 p(t) |y(t)| \, dt + \int_0^1 p(t) \, dt \]

\[ \text{(2.8)} \]

ii) Let \( y(t) > 1/n \) for \( t \in (0,1) \) and \( \gamma > 1 \) then

\[ |y| = \frac{(\gamma + 1)^2}{4} \|y\| \int_0^1 p(t) \int_{p(s)}^t ds \, dt + (\gamma + 1) \int_0^1 p(t) y'(t) \, dt \]

\[ \text{(2.9)} \]

Here

\[ |y| = \int_0^1 py^{\gamma-1} \, dt \quad \text{and} \quad \|y\|^2 = \int_0^1 py^{\gamma-1} (y')^2 \, dt \] [10].

3 Existence Theorems

In this part, we give two theorems for the existence of the solution to a two points boundary value problem which has singularities which appear both from independent and dependent variables.

It will be shown that (2.6) has a \( C[0,1] \cap C^1(0,1] \cap C^2(0,1) \) solution. To do this, the following “approximate” problems will be first established.
Then to show that (2.6) has a solution, we let $n \to \infty$; the key idea in this step is the Arzela - Ascoli theorem.

**Theorem 3.1**

Suppose (2.3), (2.4) and (2.5) are satisfied and $f : [0,1] \times (0, \infty) \times (-\infty, \infty) \to (0, \infty)$ is continuous with $\lim_{y \to 0^+} f(t, y, py') = +\infty$ uniformly on compact subset of $[0,1] \times (-\infty, \infty)$.

\[
0 < f(t, y(t), p(t)y'(t)) \leq Ay + h(y) + By^\tau + g(y, y') \text{ on } (0,1) \times (0, \infty) \times (-\infty, \infty).
\]

Here $A \geq 0$ $B \geq 0$, $\gamma \geq 0$ are constants with $h \geq 0$ on $(0, \infty)$, $g \geq 0$ on $(0,1) \times (0, \infty) \times (-\infty, \infty)$ and $\sup(p(t)) = 1$ on $[0,1]$.

And

\[
y^\tau h(y) \leq Cy^\tau + D \quad \text{and} \quad yg(y, y') \leq E(y')^2 + F|y'| \quad \text{for} \quad y > 0
\]

holding. Then a $C[0,1] \cap C^1(0,1) \cap C^2(0,1)$ solution of (3.1)$^n$ exists for each $n \in \mathbb{N}^+$ in each of the following cases:

**i)** $0 \leq \gamma \leq 1$ with

\[
A \int_0^1 p(t) \left( \int_s^1 \frac{ds}{p(s)} \right) \frac{1-\gamma}{2} dt < 1, \quad \int_0^1 p(t) \left( \int_s^1 \frac{ds}{p(s)} \right) \frac{1-\gamma}{2} dt < \infty, \quad \int_0^1 \frac{1}{p(t)} \left( \int_s^1 \frac{dz}{p(z)} \right) \frac{1-\gamma}{2} dsdt < \infty.
\]

**ii)** $\gamma > 1$ with

\[
E + A \frac{(\gamma + 1)^2}{4} \int_0^1 p(t) \left( \int_s^1 \frac{ds}{p(s)} \right) dt < 1, \quad \int_0^1 p(t) \left( \int_s^1 \frac{ds}{p(s)} \right) \frac{\gamma+1}{2} dt < \infty, \quad \int_0^1 \frac{1}{p(t)} \left( \int_s^1 \frac{dz}{p(z)} \right) \frac{\gamma+1}{2} dsdt < \infty.
\]

**Proof:**

Consider the family of problems

\[
\begin{cases}
\frac{1}{p} (py')' + f(t, y) = 0 & 0 < t < 1 \\
\lim_{t \to 0^+} \frac{1}{p} y'(t) = 0, & y(1) = \frac{1}{n} n \in \mathbb{N}^+
\end{cases}
\]

(3.1)$^n$
\[
\begin{cases}
\frac{1}{p}(py')' + \lambda f^*(t,y) = 0, & 0 < t < 1, 0 < \lambda < 1 \\
\lim_{t \to 0^+} p(t)y'(t) = 0, & y(1) = \frac{1}{n}
\end{cases}
\] (3.4)_n^n

where \( f^* > 0 \) is any continuous extension of \( f \) from \( y \geq 1/n \). Let \( y \) be a solution to (3.4)_n^n. It can be claimed that \( y \geq 1/n \) on \([0,1]\). Since \( (py')' = -\lambda pf^*(t,y,p'y) \leq 0 \), \( py' \) is nonincreasing and \( \lim_{t \to 0^+} p(t)y'(t) = 0 \) then \( p(t)y'(t) \leq 0 \), hence \( y'(t) \leq 0 \) on \([0,1]\), thus \( y \) is nonincreasing. Since \( y(1) = 1/n \), we obtain \( y(1) = 1/n \leq y(t) \leq y(0) \).

Thus \( y(t) \geq 1/n \) on \([0,1]\) and so every solution of (3.4)_n^n is also a solution of

\[
\begin{cases}
\frac{1}{p}(py')' + \lambda f(t,y) = 0, & 0 < t < 1, 0 < \lambda < 1 \\
\lim_{t \to 0^+} p(t)y'(t) = 0, & y(1) = \frac{1}{n}
\end{cases}
\] (3.5)_n^n

Now, it will be shown that there exists a constant \( M_o \), independent of \( \lambda \), (and in fact \( n \)) with

\[
\frac{1}{n} \leq y(t) \leq M_o, \quad t \in [0,1]
\] (3.6)

for each solution, \( y \in C[0,1] \cap C^1(0,1] \cap C^2(0,1) \) to (3.4)_n^n. Consider \( 0 \leq \gamma \leq 1 \) and \( \gamma > 1 \) separately.

\textbf{i)} \( 0 \leq \gamma \leq 1 \)

Assumptions (3.2) and (3.3) imply

\[ -(py')'y \leq (B + D)p^{1-\gamma} + Cpy^{1+\gamma} + Apy^2 + Ep'(y')^2 + Fp|y'| \]

Integrating from 0 to 1 yields

\[
\int_0^1 p|y'|^2 dt \leq (B + D)\int_0^1 p^1|y|^{1-\gamma} dt + C\int_0^1 p^1|y|^{1+\gamma} dt + A\int_0^1 p|y|^2 dt
\]

\[ + E\int_0^1 p|y'|^2 dt + F\int_0^1 p|y'| dt + \frac{y'(1)p(1)}{n} \]

Because \( y'(t) \leq 0 \) on \( t \in [0,1] \), \( y'(1) \leq 0 \) it is obtain that

\[
\|y'\|^2 \leq (B + D)\int_0^1 p^1|y|^{1-\gamma} dt + C\int_0^1 p^1|y|^{1+\gamma} dt + F\int_0^1 p|y'| dt + E\|y'\|^2 + A\|y\|^2
\] (3.7)

Put (2.8) into (3.7) to obtain
\[(1 - E - A \int_0^1 p(t) \frac{ds}{p(s)} dt) \| y' \|^2 \leq (B + D) \int_0^1 p(t) y'^{1-\eta} dt + C \int_0^1 p(t) y'^{r+1} dt \]

\[+ F \int_0^1 p(t) y' dt + 2A \int_0^1 p(t) dt + A \int_0^1 p(t) dt \]

(3.8)

Also since

\[y(t) = \frac{1}{n} + \int_0^1 (-y'(s)) ds = \frac{1}{n} + \int_0^1 \sqrt{p(s)(-y'(s))} \frac{1}{\sqrt{p(s)}} ds \quad t \in (0,1)\]

by using Hölder’s inequality, it is obtained

\[|y(t)| \leq \|y'(t)\| \left( \int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} + 1 \]

(3.9)

for \( t \in (0,1) \). Thus

\[\int_0^1 p(t) |y(t)|^{1-\eta} dt \leq \int_0^1 p(t) \left( 1 + \|y'(t)\| \left( \int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \right)^{1-\eta} dt \]

Because \((a + b)^c \leq 2^c (a^c + b^c)\) for \( c > 0; \, a, b \geq 0 \) the following inequality can be obtained.

\[\int_0^1 p(t) |y(t)|^{1-\eta} dt \leq 2^{1-\eta} \int_0^1 p(t) dt + 2^{1-\eta} \|y'\|^{1-\eta} \int_0^1 p(t) \left( \int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} dt \]

(3.10)

\[\int_0^1 p(t) |y(t)|^{r+1} dt \leq 2^{r+1} \int_0^1 p(t) dt + 2^{r+1} \|y'\|^{r+1} \int_0^1 p(t) \left( \int_0^1 \frac{ds}{p(s)} \right)^{\frac{r+1}{2}} dt \]

(3.11)

\[\int_0^1 p(t) |y(t)| dt \leq \int_0^1 p(t) dt + \|y'\| \int_0^1 p(t) \left( \int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} dt \]

(3.12)

\[\int_0^1 p(t) y'(t) dt = \int_0^1 \sqrt{p(t)} |y'(t)| \sqrt{p(t)} dt \leq \left( \int_0^1 |y(t)|^q dt \right)^{\frac{1}{2}} \left( \int_0^1 p(t) dt \right)^{\frac{1}{2}} = \|y\| \left( \int_0^1 p(t) dt \right)^{\frac{1}{2}} \]

(3.13)

Substitute (3.10), (3.11), (3.12) and (3.13) into (3.8) to obtain

\[(1 - E - A \int_0^1 p(t) \frac{ds}{p(s)} dt) \| y' \|^2 \leq Q_1 \| y' \|^{1-\eta} + Q_2 \| y' \|^{r+1} + Q_3 \| y' \| + Q_4 \]

(3.14)

where

\[Q_1 = 2^{1-\eta} (B + D) \int_0^1 p(t) \left( \int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} dt, \quad Q_2 = 2^{r+1} C \int_0^1 p(t) \left( \int_0^1 \frac{ds}{p(s)} \right)^{\frac{r+1}{2}} dt \]
\[
Q_3 = 2A \int_0^1 p(t) \left( \int_{\tau} \frac{ds}{p(s)} \right)^{\frac{1}{2}} dt + F \left( \int_0^1 p(t) dt \right)^{\frac{1}{2}}, \quad Q_4 = (2^{1-\gamma} (B + D) + 2^{\gamma+1} C + 2A) \int_0^1 p(t) dt
\]

Note also if \( L \geq 0 \), \( 0 \leq k < 2 \), \( 0 < m < 1 \) are given constants, then there exists a constant \( N > 0 \) such that
\[
Lx^k \leq \frac{m}{2} x^2 + N \quad \text{for all} \ x \geq 0.
\]
Thus
\[
Q_1 \|y\|^{1-\gamma} \leq \frac{1}{2} (1 - E - A \int_0^1 p(t) \int_{\tau} \frac{ds}{p(s)} dt) \|y\|^2 + N_1
\]
\[
Q_2 \|y\|^\gamma \leq \frac{1}{2} (1 - E - A \int_0^1 p(t) \int_{\tau} \frac{ds}{p(s)} dt) \|y\|^2 + N_2
\]
\[
Q_3 \|y\| \leq \frac{1}{2} (1 - E - A \int_0^1 p(t) \int_{\tau} \frac{ds}{p(s)} dt) \|y\|^2 + N_3
\]

Then from (3.14), it is obtained
\[
\frac{1}{2} (1 - E - A \int_0^1 p(t) \int_{\tau} \frac{ds}{p(s)} dt) \|y\|^2 \leq Q_5
\]
where \( Q_5 = N_1 + N_2 + N_3 + Q_4 \).

Consequently there exists a constant \( M_1 \), independent of \( \lambda \), with \( \|y\| \leq M_1 \).

From the differential equation and assumptions (3.2) and (3.3) imply
\[-(p'y')y \leq (B + D)p'y^{1-\gamma} + Cpy^{\gamma+1} + Apy^2 + Ep(y')^2 + Fp\|y'\|\]

Integrating from 0 to t yields
\[-py'y' + \int_0^t p|y'|^2 ds \leq (B + D)\int_0^t p|y|^{1-\gamma} ds + C\int_0^t p|y|^{\gamma+1} ds + \int_0^t A\int_0^t p|y|^2 ds\]
\[+ E\int_0^t p|y|^2 ds + F\int_0^t p|y'| ds\]

Because \((1 - E) > 0\) then
\[-yy' \leq (B + D)\frac{1}{p} \int_0^t p|y|^{1-\gamma} ds + C\int_0^t p|y|^{\gamma+1} ds + \int_0^t A\int_0^t p|y|^2 ds + F\int_0^t p|y'| ds\]

where
\[\int_0^t p|y(s)|^{1-\gamma} ds \leq 2^{1-\gamma} \int_0^t p(s) ds + 2^{1-\gamma} \|y\|^{1-\gamma} \int_0^t p(s) \left( \int_0^t \frac{dz}{p(z)} \right)^{\frac{1}{2}} ds\]
\[\int_0^t p|y(s)|^{\gamma+1} ds \leq 2^{\gamma+1} \int_0^t p(s) ds + 2^{\gamma+1} \|y\|^{\gamma+1} \int_0^t p(s) \left( \int_0^t \frac{dz}{p(z)} \right)^{\frac{1}{2}} ds\]
\[
\int_0^1 p|y(s)|^2 \, ds \leq 2 \int_0^1 p(s) \, ds + 2\|y'\|^2 \int_0^1 p(s) \left( \int_s^1 \frac{dz}{p(z)} \right) \, ds
\]
\[
\int_0^1 p|y'(t)| \, ds = \int_0^1 \sqrt{p} |y'(s)| \sqrt{p} \, ds \leq \left( \int_0^1 p|y'|^2 \, dt \right)^{1/2} \left( \int_0^1 p \, ds \right)^{1/2} = \|y'\| \left( \int_0^1 p \, ds \right)^{1/2}
\]

Again integrating from 0 to 1 then
\[
y^2(0) \leq 2^{1-\gamma} (B + D) \|y'\|^{1-\gamma} \int_0^1 \frac{1}{p(t)} \int_0^1 p(s) \left( \int_s^1 \frac{dz}{p(z)} \right)^{1-\gamma} \, ds dt + F \|y\|^\gamma \int_0^1 \frac{1}{p(t)} \int_0^1 p(s) \left( \int_s^1 \frac{dz}{p(z)} \right)^{1-\gamma} \, ds dt
\]
\[
+ 2^{1+\gamma} C \|y\|^\gamma+1 \int_0^1 \frac{1}{p(t)} \int_0^1 p(s) \left( \int_s^1 \frac{dz}{p(z)} \right)^{\gamma+1} \, ds dt + 2A \|y\|^\gamma \int_0^1 \frac{1}{p(t)} \int_0^1 p(s) \left( \int_s^1 \frac{dz}{p(z)} \right) \, ds dt
\]
\[
+ (2^{1-\gamma} (B + D) + 2^{1+\gamma} C + 2A) \int_0^1 \frac{1}{p(t)} \int_0^1 p(s) \, ds dt + 1 \leq M_0
\]

Consequently it is obtained that \( \frac{1}{n} \leq y(t) \leq M_0 \quad t \in [0,1] \).

ii) \( \gamma > 1 \)

Assumptions (3.2) and (3.3) imply
\[
-(py')^\gamma \leq (B + D) p + Cp y^{\gamma-\gamma} + Apy^{\gamma} + Ep|y|^2 y^{\gamma-1} + Fpy^{\gamma-1} |y'|
\]
Integrating from 0 to 1 yields
\[
y^{\gamma} \int_0^1 py^{\gamma-1} |y'|^2 \, dt \leq (B + D) \int_0^1 p dt + C \int_0^1 py^{\gamma} \, dt + A \int_0^1 py^{\gamma} \, dt
\]
\[
+ E \int_0^1 p y^{\gamma-2} |y'|^2 \, dt + F \int_0^1 p y^{\gamma-1} |y'| \, dt + \frac{y'(1)p(1)}{n^\gamma}
\]
Because \( y'(t) \leq 0 \) on \( t \in [0,1] \), \( y'(1) \leq 0 \), then the following equation can be obtained.
\[
(y - E) \|y\| \leq (B + D) \int_0^1 p dt + C \int_0^1 py^{\gamma} \, dt + A \|y\| + F \int_0^1 py^{\gamma-1} |y'| \, dt \quad (3.15)
\]
Put (2.11) into (3.15) to obtain
\[
(y - E - A \frac{(y + 1)^2}{4}) \int_0^1 \frac{ds}{p(s)} \, dt \|y\| \leq (B + D) \int_0^1 p dt + C \int_0^1 py^{\gamma} \, dt
\]
\[
+ F \int_0^1 py^{\gamma-1} |y'| \, dt + A(y + 1) \int_0^1 py^{\gamma} \, dt \quad (3.16)
\]
Also since
by using Hölder’s inequality, it is obtained

\[ y^{\frac{r+1}{2}} \leq 1 + \frac{(y + 1)}{2} \left( \frac{1}{p(s)} \right)^{\frac{1}{r}} \left( \int \frac{ds}{p(s)} \right)^{\frac{1}{r^{2}}} \]

Thus

\[
\begin{align*}
y^{r+1} & \leq 2 + \frac{(y + 1)^2}{2} \left\| y \right\|_p \left( \int p(s) \right)^{\frac{1}{r}} \left( \int \frac{ds}{p(s)} \right) \frac{y^r}{r^2} \\
y^r & \leq 2^{\frac{2y}{r+1}} + (y + 1)^{\frac{2y}{r+1}} \left\| y \right\|_p \left( \int p(s) \right)^{\frac{1}{r}} \left( \int \frac{ds}{p(s)} \right) \frac{y^r}{r^2} \\
y^{r+\tau} & \leq 2^{\frac{2(y+\tau)}{r+1}} + (y + 1)^{\frac{2(y+\tau)}{r+1}} \left\| y \right\|_p \left( \int p(s) \right)^{\frac{1}{r}} \left( \int \frac{ds}{p(s)} \right) \frac{y^r}{r^2} \\
y^{r-1} & \leq 2^{\frac{2(y-1)}{r+1}} + (y + 1)^{\frac{2(y-1)}{r+1}} \left\| y \right\|_p \left( \int p(s) \right)^{\frac{1}{r}} \left( \int \frac{ds}{p(s)} \right) \frac{y^r}{r^2}
\end{align*}
\]

By using Hölder’s inequality and \((a + b)^c \leq 2^c (a^c + b^c)\) for \(c > 0\); \(a, b \geq 0\)

\[
\begin{align*}
&\int_0^1 p y^{r-1} |y'| dt \leq \left\| y \right\|_p \left( \int_0^1 p y^{r-1} dt \right) \frac{1}{\left( \int_0^1 y^{r-1} dt \right)^{\frac{1}{2}}} \\
&\int_0^1 p y(t)^r dt \leq 2^{\frac{2y}{r+1}} \int_0^1 p(t) dt + (y + 1)^{\frac{2y}{r+1}} \left\| y \right\|_p \left( \int_0^1 \frac{ds}{p(s)} \right) \frac{y^r}{r^2} dt \\
&\int_0^1 p y^{r+\tau} dt \leq 2^{\frac{2(y+\tau)}{r+1}} \int_0^1 p(t) dt + (y + 1)^{\frac{2(y+\tau)}{r+1}} \left\| y \right\|_p \left( \int_0^1 \frac{ds}{p(s)} \right) \frac{y^{r+\tau}}{r^2} dt \\
&\int_0^1 p y^{r-1} dt \leq 2^{\frac{2(y-1)}{r+1}} \int_0^1 p(t) dt + 2^{\frac{2(y+1)^{\frac{2(y-1)}{r+1}} \left\| y \right\|_p \left( \int_0^1 \frac{ds}{p(s)} \right) \frac{y^{r-1}}{r^2} dt}
\end{align*}
\]

Substitute (3.18) into (3.16) to obtain
\[
(\gamma - E - A \frac{(\gamma + 1)^2}{4} \int_0^1 p(t) \left( \int_0^t \frac{ds}{\gamma + 1} \right)^\frac{\gamma + 1}{\gamma + 1} dt \|y\|_{\gamma, y} \leq Q_1 (\|y\|_{\gamma, y})^{\gamma + 1} + Q_2 (\|y\|_{\gamma, y})^{\gamma + 1} + Q_3 (\|y\|_{\gamma, y})^{\gamma + 1} + Q_4
\]

where

\[
Q_1 = C(\gamma + 1)^{\frac{2(\gamma + r)}{\gamma + 1}} \int_0^1 p(t) \left( \int_0^t \frac{ds}{\gamma + 1} \right)^\frac{\gamma + 1}{\gamma + 1} dt
\]

\[
Q_2 = A(\gamma + 1)^{\frac{3(\gamma + 1)}{\gamma + 1}} \int_0^1 p(t) \left( \int_0^t \frac{ds}{\gamma + 1} \right)^\frac{\gamma + 1}{\gamma + 1} dt + F^2 (\gamma + 1)^{\frac{1}{\gamma + 1}} \left( \int_0^1 p(t) \left( \int_0^t \frac{ds}{\gamma + 1} \right)^\frac{\gamma + 1}{\gamma + 1} dt \right)^\frac{1}{2}
\]

\[
Q_3 = F^2 (\gamma + 1)^{\frac{2(\gamma + 1)}{\gamma + 1}} \int_0^1 p(t) dt \left( \int_0^t \frac{dt}{\gamma + 1} \right)^\frac{1}{2}
\]

\[
Q_4 = (B + D + 2) (\gamma + 1)^{\frac{2(\gamma + r)}{\gamma + 1}} \int_0^1 p(t) dt
\]

Thus there exists a constant \( Q_5 \) with

\[
\frac{1}{2} (\gamma - E - A \frac{(\gamma + 1)^2}{4} \int_0^1 p(t) \left( \int_0^t \frac{ds}{\gamma + 1} \right)^\frac{\gamma + 1}{\gamma + 1} dt \|y\|_{\gamma, y} \leq Q_5.
\]

Hence there exists a constant \( M_1 \), independent of \( \lambda \), with

\[
\|y\|_{\gamma, y} = \int_0^1 p y^{\gamma - 1} |y|^{2} dt \leq M_1.
\]

From the differential equation and assumptions (3.2) and (3.3) imply

\[-(py')' y' \leq (B + D) p + Cpy^{\gamma + r} + Ap'y^{\gamma + r} + Ep'y^{2\gamma - 1} + Fpy^{\gamma - 1} |y|
\]

Integrating from 0 to \( t \) yields

\[- p y' y' + \int_0^t py^{\gamma - 1} |y'|^2 ds \leq (B + D) \int_0^t pds + C \int_0^t p y^{\gamma + r} ds + A \int_0^t p y^{\gamma + r} ds + E \int_0^t p y^{\gamma - 1} |y'|^2 ds + F \int_0^t p y^{\gamma - 1} |y'| ds
\]

Because \( (\gamma - E) > 0 \) then

\[- y' y' \leq (B + D) \frac{1}{p} \int_0^t pds + C \frac{1}{p} \int_0^t p y^{\gamma + r} ds + A \frac{1}{p} \int_0^t p y^{\gamma + r} ds + F \frac{1}{p} \int_0^t p y^{\gamma - 1} |y'| ds
\]
where
\[
\int_0^t p(y)^{r+1} ds \leq 2 \int_0^t p(s) ds + (\gamma + 1)^{\frac{2}{r+1}} \int_0^t p(s) \left( \int_0^z \frac{dz}{p(z)} \right) ds
\]
\[
\int_0^t p(y)^{r+2} ds \leq 2^{\frac{2(\gamma+1)}{r+1}} \int_0^t p(s) ds + (\gamma + 1)^{\frac{2(\gamma+1)}{r+1}} \left( \frac{1}{\gamma+1} \right)^{\frac{1}{r+1}} \int_0^t p(s) \left( \int_0^z \frac{dz}{p(z)} \right) ds
\]
\[
\int_0^t p(y)^{\gamma-1} |y'| ds \leq 2^{\frac{(\gamma-1)}{\gamma+1}} \left( \frac{1}{\gamma+1} \right)^{\frac{1}{r+1}} \left( \int_0^t p(s) ds \right)^{\frac{1}{2}} + 2^{\frac{1}{2}} (\gamma + 1)^{\frac{\gamma-1}{\gamma+1}} \left( \frac{1}{\gamma+1} \right)^{\frac{1}{r+1}} \int_0^t p(s) \left( \int_0^z \frac{dz}{p(z)} \right)^{\frac{1}{2}} ds
\]
Again integrating from 0 to 1 then
\[
y^{r+1}(0) \leq ((B + D) + 2A + 2^{\frac{2(\gamma+1)}{\gamma+1}} C) \int_0^t p(s) ds\text{dt} + 2^{\frac{(\gamma-1)}{\gamma+1}} \left( \frac{1}{\gamma+1} \right)^{\frac{1}{r+1}} \int_0^t p(s) ds\text{dt}
\]
\[
+ A \frac{(\gamma + 1)^2}{2} \left( \frac{1}{\gamma+1} \right)^{\frac{1}{r+1}} \left( \frac{1}{\gamma+1} \right)^{\frac{1}{r+1}} \int_0^t p(s) \left( \int_0^z \frac{dz}{p(z)} \right) ds\text{dt} + C (\gamma + 1)^{\frac{2(\gamma+1)}{r+1}} \left( \frac{1}{\gamma+1} \right)^{\frac{1}{r+1}} \left( \frac{1}{\gamma+1} \right)^{\frac{1}{r+1}} \int_0^t p(s) \left( \int_0^z \frac{dz}{p(z)} \right) ds\text{dt}
\]
\[
+ 2^{\frac{1}{2}} (\gamma + 1)^{\frac{\gamma-1}{\gamma+1}} \left( \frac{1}{\gamma+1} \right)^{\frac{1}{r+1}} \left( \frac{1}{\gamma+1} \right)^{\frac{1}{r+1}} \int_0^t p(s) \left( \int_0^z \frac{dz}{p(z)} \right)^{\frac{1}{2}} ds\text{dt} + 1 \leq M_0
\]
Thus
\[
\frac{1}{n} \leq y(t) \leq M_0, \quad t \in [0,1]
\]
Consequently in all cases there exists a constant $M_0$ independent of $\lambda$ with
\[
\frac{1}{n} \leq y(t) \leq M_0, \quad t \in [0,1].
\]
Now, let $-py' > \sup(p(t))M_0$ then $-y' > M_0$. Integration form 0 to 1 then $y(0)-y(1) > M_0$. Hence $y(0) > M_0$. This is a contradiction. Consequently $-py' \leq M_0$. Hence there exists a constant $M_2$ independent of $\lambda$ and also independent of $n$ such that $|py'| \leq M_2$. Thus
\[
\sup_{[0,1]} |py'| \leq M_2
\]
(3.19)
Now (3.6), (3.19) and Theorem 2.2 imply that (3.4)\(n\) has a solution \(y\) for each \(n \in \mathbb{N}^+\). In addition, since \(y \geq 1/n\) on \([0,1]\) then (3.5)\(n\) has a solution \(y \in C[0,1] \cap C^1(0,1) \cap C^2(0,1)\) for each \(n \in \mathbb{N}^+\).

**Theorem 3.2**

Suppose conditions of Theorem 3.1 are satisfied and \(\lim_{t \to 0^+} (p(t)/p'(t))\) exists. In addition suppose

\[
\begin{cases}
\text{for each constant } M > 0 \text{ and } H > 0 \text{ there exists a function } \Psi \\
\text{continuous on } [0,1] \text{ and positive on } (0,1) \text{ with } \\
f(t, y, py') \geq \Psi(t) \text{ on } (0,1) \times (0, M] \times [-H, H]
\end{cases}
\]

Then a \(C^1[0,1] \cap C^2(0,1)\) solution of (2.6) exists.

**Proof**

Theorem 3.1 implies that (3.1)\(n\) has a solution \(y_n\) for each \(n\) and there exists constants \(M_0\) and \(M_2\) independent of \(n\) such that

\[
|y_n| \leq M_0, \quad \text{for } t \in [0,1]
\]

and because \(p(t) > 0\) for \(t \in (0,1]\) then

\[
|p(t)y_n'(t)| = p(t)|y_n'(t)| \leq M_2, \quad \text{for } t \in [0,1]
\]

Hence from Corollary 2.2, it is obtained that

\[
|y_n'(t)| \leq M_3, \quad \text{for } t \in [0,1]
\]

where \(M_3\) is constant independent of \(n\). It follows that \(\{y_n\}\) is uniformly bounded and for \(t, s \in [0,1]\),

\[
|y_n(t) - y_n(s)| \leq |1 - \int_t^s y_n'(z)dz - 1 + \int_s^t y_n'(z)dz| \leq \int_s^t |y_n'(z)|dz \leq M_3|t - s| \leq M_3\delta = \varepsilon
\]

Hence \(\{y_n\}\) is equicontinuous on \([0,1]\), so Arzela Ascoli theorem guarantees the existence of a subsequence \(\{y_n'\}\) converging uniformly on \([0,1]\) to some \(y \in C[0,1]\).

Now \(y(1) = 0\) with \(y \geq 0\) on \([0,1]\) but in fact (3.20) implies \(y > 0\) on \([0,1]\); to see this note since \(M_0\) is independent of \(n'\) there exists \(\Psi(t)\) (independent of \(n'\)) such that

\[
-(py_n')' \geq p\Psi
\]

and consequently
\[ y_{n'}(t) \geq \frac{1}{n'} + \int_{0}^{t} \frac{1}{p(u)} \int_{0}^{u} p(s)\Psi'(s)dsdu \]

Letting \( n' \to \infty \) yields the result. Also \( y_{n'} \) satisfies the integral equation

\[ y_{n'}(t) = y_{n'}(0) + \int_{0}^{t} \frac{1}{p(u)} \int_{0}^{u} p(s)f(s, y_{n'}(s), p(s)y_{n'}(s))dsdu \]

For \( t \in [0,1] \) and \( s \in [0,t] \), we have \( f(s, y_{n'}(s), p(s)y_{n'}(s)) \to f(s, y(s), p(s)y'(s)) \) uniformly since \( f \) is uniformly continuous on compact subset of \([0,1] \times (0,M_0] \times [-M_2, M_2]\). Thus letting \( n' \to \infty \) yields

\[ y(t) = y(0) - \int_{0}^{t} \frac{1}{p(u)} \int_{0}^{u} p(s)f(s, y(s), p(s)y'(s))dsdu \]

From the integral equation, we see that

\[ y \in C^1[0,1] \cap C^2(0,1), \quad \frac{1}{p} (py')' + f(t, y, py') = 0, \quad \text{and} \quad y(1) = \lim_{t \to 0^+} p(t)y'(t) = 0 \]

4 Conclusion

In this study, the existence of solutions of singular nonlinear second-order boundary value problem (*) with the given boundary conditions has been investigated. By the proof of the Theorem 3.1 and Theorem 3.2, it is showed that there is a solution \( y \in C^1[0,1] \cap C^2(0,1) \) for the singular non-linear second-order boundary value problems.

References