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An Inequality of Fejer-Riesz Type

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Abstract
In this paper, we obtain an extension of an integral inequality of Fejer-Riesz type.
Key words: *Hardy Spaces, Inequalities*.

Özet
Bu çalışmada Fejer-Riesz tipinde bir integral eşitsizliğinin bir genellemesini elde edeceğiz.
Anahtar Kelimeler: *Hardy Uzayları, Eşitsizlikler*.

1. INTRODUCTION

Throughout let Δ be the open unit disk and let $\partial\Delta$ be the boundary of Δ. For $1 \leq p < \infty$, $H^p(\Delta)$ is the set of all functions f analytic on Δ such that

$$\lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta < \infty.$$

$$\|f\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| f(e^{i\theta}) \right|^p d\theta \right\}^{1/p} = \lim_{r\to 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right\}^{1/p}$$

defines a complete norm on $H^p(\Delta)$. In the case of $p=2$, $H^2(\Delta)$ is the class of power series $\sum_{n=0}^{\infty} a_n z^n$ with $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ which means $\{a_n\}_{n=0}^{\infty} \in \ell^2$. In this case, for $f \in H^2(\Delta)$ the norm is given by $\|f\|_2 = \left\{ \sum_{n=0}^{\infty} |a_n|^2 \right\}^{1/2}$. For more information on this spaces see [1 and 3].

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The following interesting inequality is given in [1, page 46].

LEMMA 1.1 (Fejer-Riesz Inequality). If $f \in H^p(\Delta)$ $(1 \le p < \infty)$, then the integral of $|f(x)|^p$ along the segment $-1 \le x \le 1$ converges, and

$$\int_{-1}^{1} |f(x)|^p \, dx \le \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \le \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta.$$

We shall prove an extension for $p = 2$, below.

THEOREM 1.2 Let γ be a circular arc (or a straight-line segment) satisfying $\gamma \subseteq \overline{\Delta}$. Then for every $f \in H^2(\Delta)$,

$$\frac{1}{2\pi} \int_\gamma |f(z)|^2 |dz| \le \|f\|^2_{H^2(\Delta)} = \frac{1}{2\pi} \int_{\partial\Delta} |f(z)|^2 |dz| \qquad (1.1)$$

where $|dz|$ denote the arclength measure.

Let $\mathbb{R}$ and $\mathbb{C}$ denote the real line and the complex plane respectively. Suppose $D' \subseteq \overline{\mathbb{C}} = \mathbb{C} \cup \infty$ is a simply connected domain. Then there is a canonical Hilbert Space $E^2(D')$ of analytic functions on D'. These spaces are discussed in detail in chapter 10 of [1] and the precise definition will be recalled in the next section; so these spaces will be taken for granted for the moment. The following is an immediate consequence of above theorem.

COROLLARY 1.3 Suppose that D is a disc or a codisc or a half-plane and $\gamma' \subseteq \overline{D}$ be a circular arc (or a straight line) then for every $g \in E^2(D)$,

$$\frac{1}{2\pi} \int_{\gamma'} |g(z)|^2 |dz| \le \|g\|^2_{E^2(D)} = \frac{1}{2\pi} \int_{\partial D} |g(z)|^2 |dz|$$

where ∂D denote the boundary of D.

2. PRELIMINARIES

Let D be a simply connected domain in $\overline{\mathbb{C}}$ and let φ be a Riemann mapping function for D, that is, a conformal map of D onto Δ. An analytic function g on D is said to be of class $E^2(D)$ if there exists a function $f \in H^2(\Delta)$ such that

$$g(z) \equiv f(\varphi(z))\varphi'(z)^{\frac{1}{2}} \quad (z \in D)$$

where $\varphi'^{\frac{1}{2}}$ is a branch of the square root of φ'. We define

$\|g\|_{E^2(D)} = \|f\|_{H^2(\Delta)}$. Thus, by construction, $E^2(D)$ is a Hilbert space with

$$\langle g_1, g_2 \rangle_{E^2(D)} = \langle f_1, f_2 \rangle_{H^2(\Delta)}$$

where $g_i(z) = f_i(\varphi(z))\varphi'(z)^{\frac{1}{2}}$, ($i = 1, 2$) and the map $U_\varphi : H^2(\Delta) \to E^2(D)$ given by

$$U_\varphi f(z) = f(\varphi(z))\varphi'(z)^{1/2} \quad (f \in H^2(\Delta), z \in D)$$

is an isometric bijection. If ∂D is a rectifiable Jordan curve then the same formula

$$V_\varphi f(z) = f(\varphi(z))\varphi'(z)^{1/2} \quad (f \in L^2(\partial\Delta), z \in \partial D)$$

defines an isometric bijection V_φ of $L^2(\partial\Delta)$ onto $L^2(\partial D)$, the L^2 space of normalized arc length measure on ∂D. The inverse

$$V_\psi = V_\varphi^{-1} : L^2(\partial D) \to L^2(\partial\Delta)$$

of V_φ is given by

$$V_\psi g(w) = g(\psi(w))\psi'(w)^{1/2} \quad (g \in L^2(\partial D), w \in \partial\Delta, \psi = \varphi^{-1}).$$

We recall some definition and remark.

REMARK 2.1 Suppose f is a function on an interval I in $\mathbb{R}$ and $a, b \in I$. If $(\log f)'' > 0$ we say that f is log-convex. Then we have

i) if f is log-convex then $f((1-\lambda)a + \lambda b) \le f(a)^{(1-\lambda)} f(b)^\lambda$ $(0 < \lambda < 1)$

ii) f is log-convex if and only if $(f')^2 \le ff''$.

REMARK 2.2 Suppose that $G \in L^2(\mathbb{R})$. The Fourier Transform of G is the function $\hat{G}$ given by

$$\hat{G}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(u)e^{ixu}du$$

and G is given by

$$G(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}(x)e^{-ixu}dx.$$

The following equality

$$\int_{-\infty}^{\infty} |G(u)|^2 du = \int_{-\infty}^{\infty} |\hat{G}(x)|^2 dx$$

(i.e. $\|G\|^2 = \|\hat{G}\|^2$) is called Parsevals identity. If $G \in L^2(\mathbb{R})$ we have the equality

$$\|\hat{G}\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} G(u) e^{ixu} du \right|^2 dx .$$

(A proof may be found in Rudin [4, page 189].

3. MAIN RESULT

In this section we shall prove the theorem which is mentioned in the introduction.

Proof of Theorem 1.2:

When $f \in H^2(\Delta)$, we shall write $I_\gamma = \frac{1}{2\pi} \int_{\partial\Delta} |f(z)|^2 |dz|$.

Special cases;

i) The case $\gamma \subseteq \partial\Delta$ is trivial.

ii) Suppose that $\gamma = \{z : |z| = r\}$ so that $z = re^{i\theta}$ and $|dz| = rd\theta$ (see Figure 1).

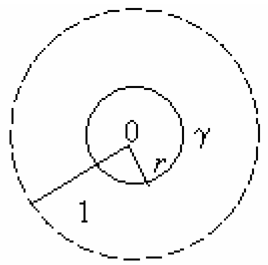

Figure 1

For $f \in H^2(\Delta)$, since

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \left(a_n \in \ell^2\right) \text{ and } \left|f(re^{i\theta})\right|^2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \overline{a_m} r^{n+m} e^{i(n-m)\theta}$$

we obtain

$$I_\gamma = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|f(re^{i\theta})\right|^2 rd\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n \overline{a_m} r^{n+m} e^{i(n-m)\theta} rd\theta$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \overline{a_m} r^{n+m} r \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)\theta} d\theta$$

$$= \sum_{n=0}^{\infty} |a_n|^2 r^{2n+1} \le \sum_{n=0}^{\infty} |a_n|^2 = \|f\|^2$$

iii) Let γ be a full circle in Δ such that $\gamma \cap \partial\Delta = \varnothing$ and suppose that $\varphi : \Delta \to \Delta$ is a conformal map and $\gamma' = \{z : |z| = r\}$ so that $z = re^{i\theta}$, $\gamma = \varphi(\gamma')$ (see Figure 2)

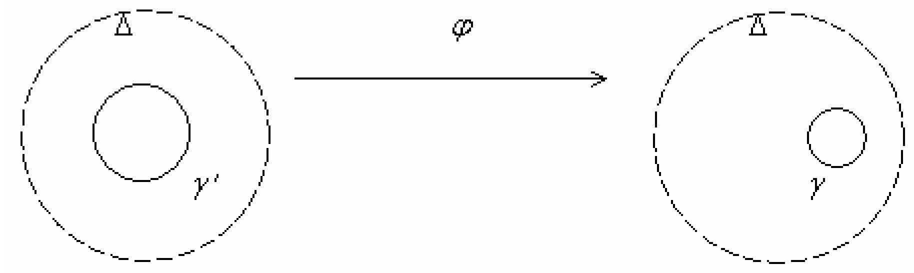

Figure 2

We know from section-2 that the formula

$$U_\varphi f(z) = f(\varphi(z))\varphi'(z)^{1/2} \quad (f \in H^2(\Delta), z \in D)$$

defines a unitary operator U_φ of $H^2(\Delta)$ onto $E^2(\Delta) = H^2(\Delta)$ so that $\|U_\varphi\| = \|f\|$ Then we have

$$I_\gamma = \frac{1}{2\pi}\int_\gamma |f(z)|^2 |dz|$$

$$= \frac{1}{2\pi}\int_{\gamma'} |f(\varphi(z))|^2 |\varphi'(z)|^2 |dz|$$

$$= \frac{1}{2\pi}\int_{\gamma'} |U_\varphi f(z)|^2 |dz|$$

$$\le \|U_\varphi f\|^2 \qquad \text{(by case ii)}$$

$$= \|f\|^2.$$

(iv) Suppose that $0 < a < 1$ and $\gamma = \{z : |z-a| = (1-a)\}$ so that

$z = a + (1-a)e^{i\theta}$ and $\gamma_n = \left\{ z : |z-a| = \frac{n}{n+1}(1-a) \right\}$ so that $z = a + \frac{n}{n+1}(1-a)e^{i\theta}$ (see Figure 3).

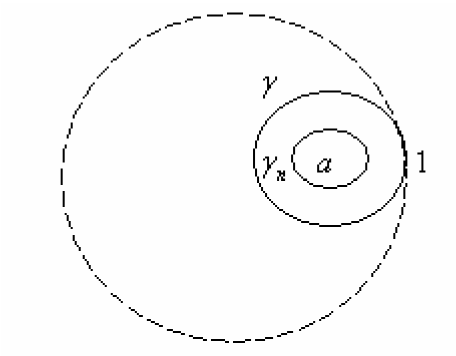

Figure 3

For each n, by case iii), it follows that

$$\frac{1}{2\pi}\int_{\gamma_n} |f(z)|^2 |dz| \le \|f\|^2 .$$

If $f \in H^2(\Delta)$ is fixed, we have

$$\frac{1}{2\pi}\int_{\gamma} |f(z)|^2 |dz| = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)d\theta$$

where

$$g(\theta) = \left| f(a+(1-a)e^{i\theta}\right|^2 (1-a)$$

and

$$\frac{1}{2\pi}\int_{\gamma_n} |f(z)|^2 |dz| = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(\theta)d\theta \le \|f\|^2$$

where

$$g_n(\theta) = \left| f(a+\frac{n}{n+1}(1-a)e^{i\theta}\right|^2 \frac{n}{n+1}(1-a).$$

Note that $(g_n) \subseteq L^1(-\pi,\pi)$ and $g_n \ge 0$ a.e. Thus by Fatou`s lemma, it follows that

$$\frac{1}{2\pi}\int_{\gamma} |f(z)|^2 |dz| = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underline{\lim}\, g_n(\theta)d\theta$$

$$\le \underline{\lim}\frac{1}{2\pi}\int_{-\pi}^{\pi} g_n(\theta)d\theta$$

$$\leq \underline{\lim}\|f\|^2 = \|f\|^2$$

(here $\underline{\lim}$ means $\lim\limits_{n\to\infty} inf$). That is, we obtain

$$\frac{1}{2\pi}\int_\gamma |f(z)|^2\,|dz| \leq \|f\|^2 .$$

v) Suppose that γ has distinct end points on $\partial\Delta$. Using a conformal map as above we can assume these end points are ±1. $\varphi(w) = \tanh(w)$ maps the infinite strip- $-\frac{\pi}{4} < v < \frac{\pi}{4}$ $(w = u + iv)$ in the w-plane onto the interior of the unit disc in the z-plane (see Figure 4).

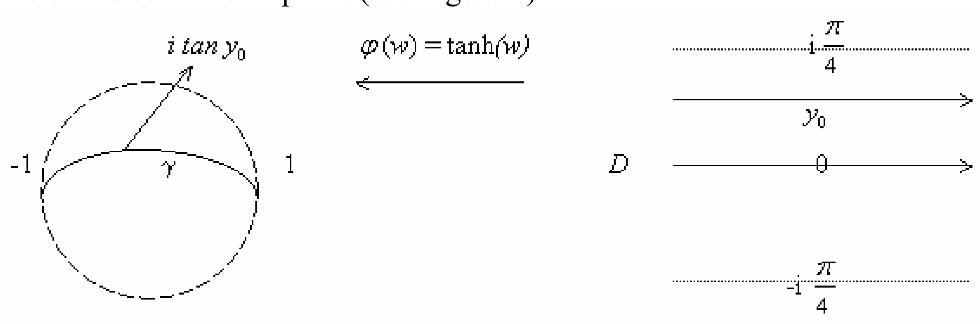

Figure 4

Suppose that $D = \{z \in \mathbb{C} : |\mathrm{Im}(w)| < \frac{\pi}{4}\}$. Then we obtain

$$E^2(D) = \left\{ g : g(w) = f(\tanh w)\sec h(w) = \sum_{n=0}^{\infty} a_n \tanh^n(w)\sec h(w) \right\}$$

where $f \in H^2(\Delta)$ and

$$I_\gamma = \frac{1}{2\pi}\int_\gamma |f(z)|^2\,|dz| = \frac{1}{2\pi}\int_{\varphi^{-1}(\gamma)} |f(\tanh(w))|^2\,|\sec h^2(w)|\,|dw|$$

(Here we used the substitution $z = \tanh w$ and the fact $dz = \sec h^2(w)dw$, $w = x + iy$, $dw = dx$), so that

$$I_\gamma = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(\tanh(x+iy))|^2\,|\sec h^2(x+iy)|\,dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |Uf(x+iy)|^2 dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |h(x+iy)|^2 dx \quad (\text{say } h = Uf \in E^2(D)) \qquad (3.1)$$

where $U : H^2(\Delta) \to E^2(D)$

$$Uf(w) = f\left(\varphi(w)\right)\varphi'(w)^{\frac{1}{2}} = f(\tanh w)\sec h(w)$$

is a unitary operator as in the case iii) so that $\|Uf\| = \|f\|$. We shall now show that if $h \in E^2(D)$ and $-\frac{\pi}{4} < y < \frac{\pi}{4}$, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|h(x+iy)|^2 dx \le \|h\|_{E^2(D)}^2$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|h(x+i\frac{\pi}{4})\right|^2 dx + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|h(x-i\frac{\pi}{4})\right|^2 dx \quad (3.2)$$

Note that if $X = \left\{h : h(z) = \sum_{n=0}^{N} a_n \tanh^n(w)\sec h(w)\right\}$, then i) X is a dense subset of $E^2(D)$ and ii) each $h \in X$ is analytic for w satisfying $|\mathrm{Im}(w)| < \frac{\pi}{4}$, in fact, this is true for w satisfying $|\mathrm{Im}(w)| < \frac{\pi}{2}$. Suppose that $h \in X$. By (3.1)

$$I_y = \frac{1}{2\pi}\int_{-\infty}^{\infty}|h(x+iy)|^2 dx, \quad -\frac{\pi}{4} < y < \frac{\pi}{4}$$

so, by (3.2) we need to show that $I_y \le I_{\frac{\pi}{4}} + I_{\frac{-\pi}{4}}$. We will show that $\left(I_y{}'\right)^2 \le I_y I_y{}''$. For $h \in X$ there is a $g \in L^2(\mathbb{R})$ such that $h(z) = \int_{-\infty}^{\infty} g(u)e^{izu}du$ (this is the Paley-Wiener theorem, see [1, page 196 and 2, page 132]). If we set $G(u) = g(u)e^{-yu}$, $-\frac{\pi}{4} < y < \frac{\pi}{4}$ then the Fourier Transform $\hat{G}$ of G is given by $\hat{G}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(u)e^{ixu}du,$ but also we have $G(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{G}(x)e^{-ixu}dx$ and $\left\|\hat{G}\right\|^2 = \|G\|^2$. For $-\frac{\pi}{2} < y < \frac{\pi}{2}$ we obtain

$$I_y = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} g(u)e^{i(x+iy)u}du\right|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} g(u)e^{-yu}e^{ixu}du\right|^2 dx$$

$$= \frac{1}{2\pi}\left\|\hat{G}\right\|^2 = \frac{1}{2\pi}\|G\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|g(u)e^{-yu}\right|^2 du$$

and consequently

$$I_y = \frac{1}{2\pi}\int_{-\infty}^{\infty}|g(u)|^2 e^{-2yu}du$$

$$I_y\,' = \frac{1}{2\pi}\int_{-\infty}^{\infty} -2u\left|g(u)\right|^2 e^{-2yu}\,du$$
$$I_y\,'' = \frac{1}{2\pi}\int_{-\infty}^{\infty} -4u^2\left|g(u)\right|^2 e^{-2yu}\,du$$

In view of above equalities, it follows by Schwarz Inequality that $\left(I_y\,'\right)^2 \le I_y I_y\,''$. Thus I_y is log-convex. So from Remark 2.1, $I_{(1-\lambda)a+\lambda b} \le I_a^{(1-\lambda)} I_b^{\lambda}$ $(0<\lambda<1)$. Note that if $\alpha,\beta>0$ and $\alpha+\beta=1$ and $x,y>0$ then $x^{\alpha}y^{\beta} \le x+y$. Hence for $h\in X$

$$\begin{aligned} I_y &= I_{(1-\lambda)\frac{\pi}{4}+\lambda\left(\frac{-\pi}{4}\right)} \qquad \left(\frac{-\pi}{4}<y<\frac{\pi}{4}\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|h(x+iy)\right|^2 dx \le I_{\frac{\pi}{4}}^{\alpha} + I_{\frac{-\pi}{4}}^{\beta} \\ &\le I_{\frac{\pi}{4}} + I_{\frac{-\pi}{4}}. \end{aligned} \tag{3.3}$$

We shall finish the proof of this case by showing that the inequality

$$I_y = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|h(x+iy)\right|^2 dx \le \left\|h\right\|_{E^2(D)}^2 \qquad \left(\frac{-\pi}{4}<y<\frac{\pi}{4}\right)$$

is true for all $h\in E^2(D)$. Now suppose that $h_N(z)=\sum_{n=0}^{N} a_n \tanh^n(z)\sec h(z)$; that is, $h_N\in X$, for $N=0,1,2,...$ and $h(z)=\sum_{n=0}^{\infty} a_n \tanh^n(z)\sec h(z)$, i.e., $h\in E^2(D)$. Then

$$h_N \to h \qquad \left(\text{means } \int_{\partial D}\left|h_N(z)-h(z)\right|\left|dz\right| \to 0\right).$$

From (3.3), the following inequality

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|h_N(x+iy)\right|^2 dx \le \left\|h_N\right\|_{E^2(D)}^2$$

holds. By Fatou`s Lemma

$$\int_{-\infty}^{\infty}\underline{\lim}\left|h_N(x+iy)\right|^2 dx \le \underline{\lim}\int_{-\infty}^{\infty}\left|h_N(x+iy)\right|^2 dx$$

so

$$\begin{aligned} I_y &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|h(x+iy)\right|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underline{\lim}\left|h_N(x+iy)\right|^2 dx \\ &\le \underline{\lim}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|h_N(x+iy)\right|^2 dx \end{aligned}$$

$$\leq \underline{\lim} \|h_N\|^2_{E^2(D)} = \|h\|^2_{E^2(D)}.$$

Hence for all $h \in E^2(D)$, we have the inequality

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |h(x+iy)|^2 dx \leq \|h\|^2_{E^2(D)}$$

as required. We now verified (1.1) in all cases.

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