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Para Hamiltonian Equations With Poisson Brackets

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Abstract

In this study, taking care of the framework of para-Kählerian manifolds it was introduced para-complex analogue of Hamiltonian equations with Poisson bracket. Also, it was shown that a canonical transformation preserves para-complex Hamiltonian equations with Poisson structure. Finally, the geometrical and mechanical conclusions on the para-mechanic systems have been suggested.

Keywords: *para-Kählerian manifolds; Hamiltonian systems; Poisson manifold; canonical transformation*

Özet

Bu çalışmada, para-Kähler manifoldların çatısı dikkate alınarak, Poisson parantezli Hamilton denklemlerinin para-kompleks benzeri tanıtıldı. Aynı zamanda; bir kanonik dönüşümün Poisson yapılı para-kompleks Hamilton denklemleri koruduğu gösterildi. Sonuçta ise; para-mekanik sistemler üzerindeki geometrik ve mekanik sonuçlar tartışıldı.

Anahtar Kelimeler*: para- Kähler manifoldlar; Hamilton sistemler; Poisson manifold; kanonik dönüşüm*

1. Introduction

Differential geometry provides a good framework for studying Hamiltonian formalisms of classical mechanics. It is possible to show in [1,2,3] some numerous articles and books where differential geometric methods in mechanics are presented. In

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fact, we may say that the role of symplectic geometry in Hamiltonian theories is similar to that of tangent geometry in Lagrangian theories [3]. The dynamics of Hamiltonian formalisms is characterized by a suitable vector field defined on cotangent bundles which are phase-spaces of momentum of a given configuration manifold **Q** of dimension **m**. If $\mathbf{H}:\mathbf{T^*Q}\rightarrow\mathbf{R}$ is a regular Hamiltonian function then there is a unique vector field $\mathbf{Z_H}$ on $\mathbf{T^*Q}$ such that dynamical equations

$$i\_\mathbf{Z_H}\, \Phi = \Phi\,(\mathbf{Z_H}) = \mathbf{dH}, \qquad (1)$$

where **Φ** is the symplectic form and **H** stands for Hamiltonian function. The paths of the Hamiltonian vector field $\mathbf{Z_H}$ are the solutions of the Hamiltonian equations. The triple $(\mathbf{T^*Q},\ \Phi,\ \mathbf{Z_H})$ is called Hamiltonian system on the cotangent bundle **T*Q** with symplectic form **Φ.** Hamilton equations with Poisson structure are written as

$$\mathbf{dq^i\,/dt=\{q^i,H\}}\ , \quad \mathbf{dp_i\,/dt=\{p_i,H\}}, \qquad (2)$$

where $(\mathbf{q^i}, \mathbf{p_i})$, $1\leq \mathbf{i}\leq m$ are canonical coordinates on **T*Q.**

Complex (para-complex) analogues of the Hamiltonian equations were obtained in the framework of Kählerian (para-Kählerian) manifolds and the geometric results on a complex (para-complex) mechanical systems were found [4,5]. Also, complex version of Hamiltonian equations with Poisson structure was introduced [6].

The goal of this study is to make a contribution to the modern development of Hamiltonian formalisms of classical mechanics in terms of differential-geometric methods on differentiable manifolds. From this point of view, this manuscript presents the para-complex analogues of Hamiltonian equations with Poisson bracket and discusses geometrical and mechanical conclusions on a para-mechanic systems.

The present paper is structured as follows. In section 2, it is recalled para-complex and para-Kählerian manifolds, and also para-complex analogues of Hamiltonian equations. In section 3, Poisson structure is generalized to para-Kählerian manifolds. In section 4, it is obtained para-complex version of Hamiltonian equations with Poisson bracket. In the conclusion section, geometrical and mechanical conclusions on Hamiltonian mechanics systems were suggested.

2. Preliminaries

In this study, all manifolds and geometric objects are differentiable and the Einstein summation convention is in use. Also, it is denoted by **A** the set of para-complex numbers, by ***F*****(TM)** the set of para-complex functions on **TM**, by $\mathfrak{X}$**(TM)** the set of para-complex vector fields on **TM** and by Λ^{1}(**TM**) the set of para-complex 1-forms on **TM**. The definitions and geometric structures on the differential manifold **M** given by [7] may be extended to **TM** as follows:

2.1 Para-complex Manifolds

Definition 1: A tensor field **J** of type (1,1) on **TM** such that $J^2=I$ is called an almost product structure on a tangent bundle **TM** of configuration manifold **M** of real dimension **m.** The pair (**TM**,**J**) is said to be an almost product manifold. An almost para-complex manifold is an almost product manifold (**TM**,**J**) such that the two eigenbundles $TT^{+}M$ and $TT^{-}M$ associated to the eigenvalues +1 and -1 of **J**, respectively have the same rank. The dimension of an almost para-complex manifold is necessarily even. Equivalently, a splitting of the tangent bundle **TTM** of tangent bundle **TM**, into the Whitney sum of two sub bundles on $TT^{\pm}M$ of the same fibre dimension is called an almost para-complex structure on **TM**. An almost para-complex structure on a **2m**-dimensional manifold **TM** may alternatively be defined as a **G**- structure on **TM** with structural group **GL**(**n**,**R**)x **GL**(**n**,**R**).

A para-complex manifold is an almost para-complex manifold (**TM**,**J**) such that the **G**- structure defined by the tensor field **J** is integrable. Assume that x^i and (x^i, y^i), $1\leq i\leq m$ are a real coordinate system on neighbourhoods U_p and TU_p of any points **p** and T_p of **M** and **TM** and, also $\{(\partial/\partial x^i)_p, (\partial/\partial y^i)_p\}$ and $\{(dx^i)_p, (dy^i)_p\}$ natural bases over **R** of the tangent space $T_p(TM)$ and the cotangent space $T_p{}^*(TM)$ of **TM**, respectively. It can be seen to be

$$J(\partial/\partial x^i)= \partial/\partial y^i \ , \ J(\partial/\partial y^i)= \partial/\partial x^i \tag{3}$$

and

$$J^*(dx^i)=-dy^i, \ J^*(dy^i)=-dx^i. \tag{4}$$

Let $z^i=x^i+j\, y^i$ ($\acute{z}^i=x^i-j\, y^i$), $j^2=1$, $1\leq i\leq m$ be a para-complex local coordinate system on a neighbourhood TU_p of any point T_p of **TM**. We define the vector fields as:

$$J(\partial/\partial z^i)_p= \tfrac{1}{2}\{(\partial/\partial x^i)_p- j\, (\partial/\partial y^i)_p\}, \ \ J(\partial/\partial \acute{z}^i)_p= \tfrac{1}{2}\{(\partial/\partial x^i)_p+ j\, (\partial/\partial y^i)_p\}, \tag{5}$$

and the dual co-vector fields as:

$$(dz^i)_p =(dx^i)_p+ j(dy^i)_p \ , \ (d\acute{z}^i)_p =(dx^i)_p- j(dy^i)_p \, , \tag{6}$$

which represent the bases of the tangent space $\mathbf{T_p(TM)}$ and cotangent space $\mathbf{T_p}^*(\mathbf{TM})$ of **TM**, respectively. Then the following can be found

$$\mathbf{J}(\partial/\partial \mathbf{z}^{\mathbf{i}})=-\mathbf{j}(\partial/\partial \mathbf{z}^{\mathbf{i}})_{\mathbf{p}}\ ,\ \mathbf{J}(\partial/\partial \acute{\mathbf{z}}^{\mathbf{i}})_{\mathbf{p}}=\mathbf{j}(\partial/\partial \acute{\mathbf{z}}^{\mathbf{i}}). \qquad (7)$$

The dual endomorphism $\mathbf{J}^*$ of the cotangent space $\mathbf{T_p}^*(\mathbf{TM})$ at any point $\mathbf{T_p}$ of manifold **TM** satisfies $\mathbf{J}^{*2}$=I and is defined by

$$\mathbf{J}^*\ (\mathbf{dz}^{\mathbf{i}}) =-(\mathbf{dz}^{\mathbf{i}}),\ \mathbf{J}^*\ (\mathbf{d}\acute{\mathbf{z}}^{\mathbf{i}}) =(\mathbf{d}\acute{\mathbf{z}}^{\mathbf{i}}). \qquad (8)$$

2.2 Para-Kählerian Manifolds

Definition 2: An almost para-Hermitian manifold (**TM**,**g**,**J**) is a differentiable manifold **TM** endowed with an almost product structure **J** and a pseudo- Riemannian metric **g**, compatible in the sense that

$$\mathbf{g}(\mathbf{JX},\mathbf{Y})+\mathbf{g}(\mathbf{X},\mathbf{JY})=0,\ \text{for all}\ \mathbf{X},\mathbf{Y}\in \mathfrak{X}(\mathbf{TM}). \qquad (9)$$

An almost para-Hermitian structure on a differentiable manifold **TM** is **G**- structure on **TM** whose structural group is the representation of the para unitary group **U**(**n**,**A**) given in [7]. A para-Hermitian manifold is a manifold with an integrable almost para-Hermitian structure (**g**,**J**). 2-covariant skew tensor field Φ defined by Φ (**X**,**Y**)=**g**(**X**,**JY**) is called fundamental 2-form. An almost para-Hermitian manifold (**TM**,**g**,**J**), such that Φ is closed shall be called an almost para-Kählerian manifold.

A para-Hermitian manifold (**TM**,**g**,**J**) is said to be a para-Kählerian manifold if Φ is closed. Also, by means of geometric structures, one may show that (**T*M**,**g**,**J**) is a para-Kählerian manifold.

2.3 Para Hamiltonian Equations

Here, we obtain para-Hamiltonian equations for classical mechanics structured on para-Kählerian manifold **T*M.**

Let **T*M** be any para-Kählerian manifold and $\{\mathbf{z}^{\mathbf{i}},\mathbf{z}^{\mathbf{i}}\}$,$1\leq \mathbf{i} \leq m$ its para-complex coordinates. Suppose that $\{(\partial/\partial \mathbf{z}^{\mathbf{i}})_{\mathbf{p}}, (\partial/\partial \mathbf{z}^{\mathbf{i}})_{\mathbf{p}}\}$ and $\{(\mathbf{dz}^{\mathbf{i}})_{\mathbf{p}}, (\mathbf{d\ z}^{\mathbf{i}}\ )_{\mathbf{p}}\}$ are bases over the set of para-complex numbers **A** of tangent space $\mathbf{T_p(TM)}$ and the cotangent space $\mathbf{T_p}^*(\mathbf{TM})$ of **TM**. Taking care of almost para-complex structure $\mathbf{J}^*$ given by **Eq.**(**8**), para-Liouville form λ is calculated as $\lambda = \mathbf{J}^*\ (\omega)=1/2\ \mathbf{j}(\mathbf{z}^{\mathbf{i}}\mathbf{dz}^{\mathbf{i}} - \mathbf{z}^{\ \mathbf{i}}\mathbf{dz}^{\mathbf{i}})$ such that para-complex 1-form $\omega =1/2\ (\mathbf{z}^{\mathbf{i}}\ \mathbf{dz}^{\mathbf{i}} +\mathbf{z}^{\ \mathbf{i}}\mathbf{dz}^{\mathbf{i}})$ on **T*M**. If $\Phi = -\mathbf{d}\ \lambda$ is closed para-Kählerian form, then Φ is also a para-symplectic structure on $\mathbf{T^*_pM}$.

Proposition 1: Let **T*M** be para-Kählerian manifold with closed para-Kählerian form Φ. Para-Hamiltonian vector field $\mathbf{Z_H}$ on para-Kählerian manifold with closed para-Kählerian form Φ is given by

$$\mathbf{Z_H = -j\, \partial H/\partial z_i\, \partial/\partial z_i + j\, \partial H/\partial z_i\, \partial/\partial z_i}\ , 1\leq \mathbf{i}\leq m \qquad (\mathbf{10})$$

on **T*M**.

By para-Hamiltonian equations on para-Kählerian manifold **T*M,** we call the following equations:

$$\mathbf{dz_i/dt=-j\, \partial H/\partial \dot{z}_i\ ,\ d\dot{z}_i\, /dt=j\, \partial H/\partial z_i} \qquad (\mathbf{11})$$

3. Poisson Manifolds

Let **T*M** be a para-Kählerian manifold with closed para-Kählerian form Φ. If the closed para-Kählerian form Φ on **T*M** is symplectic structure, all para-Kählerian manifolds are also symplectic manifolds.

Proposition 2: Let (**T*M, Φ**) and (**S**, ω) be symplectic manifolds of same dimension and let **H** be a symplectic transformation from (**T*M,Φ**) to (**S**, ω). Then (**Th**)$\mathbf{Z_{Fof}=Z_F}$ holds for any function **F** on **S**.

Assume that (**T*M, Φ**) is a symplectic manifold. Let **F** and **G** be $\mathbf{C^\infty}$ para-complex functions on **T*M** . Then Poisson bracket of **F** and **G** is defined by

$$\{\mathbf{F},\mathbf{G}\}= \Phi\ (\mathbf{Z_F},\mathbf{Z_G}), \qquad (\mathbf{12})$$

where $\mathbf{Z_F}$, $\mathbf{Z_G}$ are Hamiltonian vector fields on **T*M** defined by $\mathbf{i\ \_Z_F\ \Phi = \Phi\ (Z_F)=dF}$ and $\mathbf{i\ \_Z_G\ \Phi = \Phi\ (Z_F)=dG}$, respectively.

Definition 3: Poisson structure is called a bilinear map defined by

$$\mathbf{C^\infty(\ T^*M)}\ x\ \mathbf{C^\infty(\ T^*M)} \rightarrow \mathbf{C^\infty(\ T^*M)} \qquad (\mathbf{13})$$
$$(F,G) \rightarrow \{F,G\}$$

on a para-Kählerian manifold **T*M** if the following identities are verified.

(**i**) (Skew symmetry) {**F**,**G**}=-{**G**,**F**},

(**ii**) (Jacobi identity) {**F**,{**G**,**H**}}+{**G**,{**H**,**F**}}+{**H**,{**F**,**G**}}=0,

(**iii**) (Leibniz rule) {**FG**,**H**}=**F**{**G**,**H**}+{**F**,**G**}**H**,

where $\mathbf{C^\infty}$(**T*M**) is the space of $\mathbf{C^\infty}$ functions on **T*M**.

Para-Kählerian manifold endowed with Poisson structure { , } is also called Poisson manifold.

Let { , } be a Poisson structure on the para-Kählerian manifold.

From (**iii**) of **Definition 3** we see that the map

$$\{ \mathbf{F}, \}: \mathbf{C}^\infty(\mathbf{T^*M}) \to \mathbf{C}^\infty(\mathbf{T^*M}), \quad (\mathbf{G}) \to \{ \mathbf{F}, \mathbf{G}\} \tag{14}$$

is a derivation. Therefore there is a unique vector field $\mathbf{Z_G}$ on **T*M** such that

$$\mathbf{Z_G}\ \mathbf{F}=\{ \mathbf{F}, \mathbf{G}\}, \tag{15}$$

where $\mathbf{Z_G}$ is said the Hamiltonian vector field of $\mathbf{C}^\infty$ para-complex function **G** on para-Kählerian manifold **T*M** with closed para-Kählerian form Φ.

4. Para Hamiltonian Equations with Poisson brackets

In this section, we obtain para-complex Hamiltonian equations on **T*M** with Poisson structure { , }. Taking **Eq**.(**10**), we deduce that the Poisson bracket of two functions **F** and **G** is

$$\{\mathbf{F},\mathbf{G}\}= -\mathbf{j}\ \partial \mathbf{F}/\partial \mathbf{z_i}\ \partial \mathbf{G}/\partial \mathbf{z'_i} + \mathbf{j}\partial \mathbf{F}/\partial \mathbf{z'_i}\ \partial \mathbf{G}/\partial \mathbf{z_i}. \tag{16}$$

From **Eq**.(**16**), we obtain the Poisson brackets of the canonical coordinates:

$$\{\mathbf{z_i},\mathbf{z_j}\}=\{\mathbf{z'_i},\mathbf{z'_j}\}=0,\ \{\mathbf{z_i},\mathbf{z'_j}\}=\delta_i^j \tag{17}$$

Furthermore, if **F** is a function on **T*M**, we get

$$\{\mathbf{F},\mathbf{z_i}\}=-\{\mathbf{z_i},\mathbf{F}\}=\mathbf{j}\partial \mathbf{F}/\partial \mathbf{z'_i}\ ,\ \{\mathbf{F}, \mathbf{z'_i}\}=-\{ \mathbf{z'_i},\mathbf{F}\}=\mathbf{j}\partial \mathbf{F}/\partial \mathbf{z_i} \tag{18}$$

Finally, using **Eqs**.(**11**) and (**18**), para-complex Hamilton equations with Poisson structure are calculated as

$$\mathbf{dz_i/dt}=-\{\mathbf{z_i},\mathbf{H}\}\ ,\ \mathbf{dz'_i\ /dt}=-\{\mathbf{z'_i},\mathbf{H}\} \tag{19}$$

Now, we have a question. We try to solve it. A canonical transformation **h**: (**T*M**, Φ) → (**T*M**, Φ) preserves para-complex Hamiltonian equations? To see this it is sufficient to show that the Poisson brackets are invariant under the action of **H**. Firstly, let us take

$$\mathbf{h}^*\{\mathbf{F},\mathbf{G}\}=\{\mathbf{F},\mathbf{G}\}\mathbf{o}\mathbf{h} \tag{20}$$

Then we have

$\{F,G\}oh=(Z_G F)oh=(((Th)Z_G oh)F)oh=(Z_G oh)(Foh)=\{Foh,Goh\}=\{h^*F,h^*G\}$, **(21)**

by means of **Eq**.(**15**) and **Proposition 2**. Hence considering **Eqs**.(**20**) and (**21**) it is

$$h^*\{F,G\}=\{h^*F,h^*G\} \text{ or } \{F,G\}oh=\{Foh,Goh\}. \quad (22)$$

Finally, Poisson brackets are seen invariant under the action of **h**.

Especially, if $h:(z_i, \acute{z}_i) \to (\check{z}_i, \check{z}_i')$, where $(z_i,\acute{z}_i)$ and $(\check{z}_i, \check{z}_i')$ are canonical coordinates on **T*M**, we have

$h^*\{z_i,H\}=\{z_i oh,Hoh\}=\{\check{z}_i,K\}=d\check{z}_i/dt,\ h^*\{\acute{z}_i,H\}=\{\acute{z}_i oh,Hoh\}=\{\check{z}_i',K\}=d\check{z}_i'/dt,$ (23)

such that **K=h*H=Hoh**. Thus we call to be Kamiltonian **K** in a canonically transformed set of coordinates. It is said that **K** is identical to **H**, with the possible exception of an arbitrary additive constant if

$$K \equiv H+d\Omega/dt, \quad (24)$$

where Ω is any function of phase space coordinates with continuous second derivatives.

Conclusion

Taking care of the considerations the above, it is clear that Poisson bracket is the most important operation given by the symplectic and/or Kählerian structure. We conclude that the Hamiltonian formalisms in generalized classical mechanics and field theory can be intrinsically characterized on the para-Kählerian manifold endowed with Poisson structure { , }. The geometric approach of para-complex Hamiltonian systems is that solutions of Hamiltonian vector field Z_H on para-Kählerian manifold T*M are paths para-complex Hamiltonian equations obtained in Eq. (19) on T*M with Poisson bracket { , }.

With respect to **Eq**. (**23**), it was shown that canonical transformations preserve the form of Hamiltonian equations. Φ being useful in Hamiltonian mechanics as well as thermodynamics given in **Eq**. (**24**) is known generating function for canonical transformation. Moreover, four types of time-dependent generating functions are possible to define. By means of these, it may obtain generalized Maxwell relations [**8**].

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